Examples

of

Differential Equations

Osborne

Florian Cajori

EXAMPLES

OF

DIFFERENTIAL EQUATIONS

WITH

RULES FOR THEIR SOLUTION.

BY

GEORGE A. OSBORNE, S.B.

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY.

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PREFACE.

THIS work has been prepared to meet a want felt by the author in a practical course on the subject, arranged for advanced students in Physics. It is intended to be used in connection with lectures on the theory of Differential Equations and the derivation of the methods of solution.

Many of the examples have been collected from standard treatises, but a considerable number have been prepared by the author to illustrate special difficulties, or to provide exercises corresponding more nearly with the abilities of average students. With few exceptions they have all been tested by use in the class-room.

G. A. OSBORNE.

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CONTENTS.

CHAPTER I.

DEFINITIONS. — DERIVATION OF THE DIFFERENTIAL EQUATION FROM THE COMPLETE PRIMITIVE.

SOLUTION OF DIFFERENTIAL EQUATIONS.

CHAPTER II.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND DEGREE BETWEEN TWO VARIABLES.

CHAPTER III.

EXACT DIFFERENTIAL EQUATIONS. — INTEGRATING FACTORS.

CHAPTER IV.

DIFFERENTIAL EQUATIONS OF FIRST ORDER AND DEGREE WITH THREE VARIABLES.

CHAPTER V.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER OF HIGHER DEGREES.

CHAPTER VI.

SINGULAR SOLUTIONS.

DIFFERENTIAL EQUATIONS OF HIGHER ORDERS.

CHAPTER VII.

LINEAR DIFFERENTIAL EQUATIONS.

CHAPTER VIII.

SPECIAL FORMS OF DIFFERENTIAL EQUATIONS OF HIGHER ORDERS.

CHAPTER IX.

SIMULTANEOUS DIFFERENTIAL EQUATIONS.

CHAPTER X.

EXAMPLES OF DIFFERENTIAL EQUATIONS.

CHAPTER I.

DEFINITIONS. DERIVATION OF THE DIFFERENTIAL EQUATION FROM THE COMPLETE PRIMITIVE.

1. A *differential equation* is an equation containing differentials or differential coefficients.

The *solution* of a differential equation is the determination of another equation free from differentials or differential coefficients, from which the former may be derived by differentiation.

The *order* of a differential equation is that of the highest differential coefficient it contains; and its *degree* is that of the highest power to which this highest differential coefficient is raised, after the equation is freed from fractions and radicals.

The solution of a differential equation requires one or more integrations, each of which introduces an arbitrary constant. The most general solution of a differential equation of the nth order contains n arbitrary constants, whatever may be its degree. This general solution is called the *complete primitive* of the given differential equation.

2. *To derive a differential equation of the first order from its complete primitive.*

Differentiate the primitive; and if the arbitrary constant has disappeared, the result is the required differential equation. If not, the elimination of this constant between the two equations will give the differential equation.

3. Form the differential equations of the first order of which the following are the complete primitives, c being the arbitrary constant:

1. $\log(xy) + x = y + c.$

2. $(1 + x^2)(1 + y^2) = cx^2.$

3. $\cos y = c \cos x.$

4. $y = ce^{-\tan^{-1}x} + \tan^{-1}x - 1.$

5. $y = (cx + \log x + 1)^{-1}.$

6. $y = cx + c - c^3.$

7. $(y + c)^2 = 4ax.$

8. $y^2 \sin^2 x + 2cy + c^2 = 0.$

9. $e^{2y} + 2cxe^y + c^2 = 0.$

4. *To derive a differential equation of the second order from its complete primitive.*

Differentiate the primitive twice successively, and eliminate, if necessary, the two arbitrary constants between the three equations.

5. Form the differential equations of the second order of which the following are the complete primitives, c_1 and c_2 being the arbitrary constants:

1. $y = c_1 \cos(ax + c_2).$

2. $y = c_1 e^{ax} + c_2 e^{-ax}.$

3. $y = (c_1 + c_2 x)e^{ax}.$

4. $y = c_1 x^3 + \dfrac{c_2}{x}.$

5. $y = c_1 \sin nx + c_2 \cos nx + \dfrac{\cos ax}{n^2 - a^2}.$

6. The preceding process may be extended to the derivation of equations of higher orders from their primitives.

7. Form the differential equations of the third order of which the following are the complete primitives:

1. $y = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^x.$

2. $ye^x = c_1 e^{2x} + c_2 \sin x\sqrt{2} + c_3 \cos x\sqrt{2}.$

3. $y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right)e^x + c_3.$

Form the differential equations of the fourth order of which the following are the complete primitives:

4. $y = (c_1 + c_2 x + c_3 x^2)e^x + c_4.$

5. $x^3 + a^4 y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \sin ax + c_4 \cos ax.$

CHAPTER II.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE BETWEEN TWO VARIABLES.

General Form, $\quad Mdx + Ndy = 0,$

where M, N, are each functions of x and y.

8. Form, $\quad XYdx + X'Y'dy = 0,$

where X, X', are functions of x alone, and Y, Y', functions of y alone.

Divide so as to separate the variables, and integrate each part separately.

9. Solve the following equations:

1. $(1+x)y\,dx + (1-y)x\,dy = 0.$

2. $(x^2 - yx^2)\dfrac{dy}{dx} + y^2 + xy^2 = 0.$

3. $\dfrac{dy}{dx} = \dfrac{1+y^2}{(1+x^2)xy}.$

4. $a\left(x\dfrac{dy}{dx} + 2y\right) = xy\dfrac{dy}{dx}.$

5. $(1+y^2)dx = (y+\sqrt{1+y^2})(1+x^2)^{\frac{3}{2}}dy.$

6. $\sin x \cos y\, dx = \cos x \sin y\, dy.$

7. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0.$

8. $\sec^2 x \tan y\, dy + \sec^2 y \tan x\, dx = 0.$

9. $\dfrac{dy}{dx} + \dfrac{1+y+y^2}{1+x+x^2} = 0.$

10. *Homogeneous equations.*

Substitute $y = vx$; in the resulting equation between v and x, the variables can be separated. (See Art. 8.)

11. Solve the following equations:

1. $(y - x)\,dy + y\,dx = 0.$

2. $(2\sqrt{xy} - x)\,dy + y\,dx = 0.$

3. $y^2 + x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}.$

4. $x\dfrac{dy}{dx} = y + \sqrt{x^2 + y^2}.$

5. $x\cos\dfrac{y}{x}\cdot\dfrac{dy}{dx} = y\cos\dfrac{y}{x} - x.$

6. $(8y + 10x)\,dx + (5y + 7x)\,dy = 0.$

7. $(x + y)\dfrac{dy}{dx} = y - x.$

8. $x\cos\dfrac{y}{x}(y\,dx + x\,dy) = y\sin\dfrac{y}{x}(x\,dy - y\,dx).$

9. $x + y\dfrac{dy}{dx} = my.$

   (1), $m < 2$; (2), $m = 2$; (3), $m > 2$.

10. $$[(x^2 - y^2)\sin\alpha + 2xy\cos\alpha - y\sqrt{x^2 + y^2}]\frac{dy}{dx}$$
$$= 2xy\sin\alpha - (x^2 - y^2)\cos\alpha + x\sqrt{x^2 + y^2}.$$

12. Form,

$$(ax + by + c)\,dx + (a'x + b'y + c')\,dy = 0.$$

Substitute $\qquad x = x' + \alpha, \quad y = y' + \beta,$

and determine the constants α, β, so that the new equation between x' and y' may be homogeneous. (See Art. 10.)

This method fails when $\frac{a}{a'}=\frac{b}{b'}\cdot$ In this case put $ax+by=z$, and obtain a new equation between x and z or between y and z; the variables can then be separated.

13. Solve the following equations:

1. $(3y-7x+7)dx+(7y-3x+3)dy=0.$

2. $(4x+2y-1)\frac{dy}{dx}+2x+y+1=0.$

3. $\frac{dy}{dx}=\frac{7y+x+2}{3x+5y+6}\cdot$

4. $(2y+x+1)\,dx=(2x+4y+3)dy.$

5. $2x-y+1+(x+y-2)\frac{dy}{dx}=0.$

14. *Linear Form*, $\frac{dy}{dx}+Py=Q,$

where P, Q, are independent of y.

Solution, $$y=e^{-\int Pdx}\left(\int Qe^{\int Pdx}dx+c\right)\cdot$$

15. Solve the following equations:

1. $x\frac{dy}{dx}-ay=x+1.$

2. $x(1-x^2)dy+(2x^2-1)y\,dx=ax^3dx.$

3. $(1-x^2)^2\frac{dy}{dx}+y\sqrt{1-x^2}=x+\sqrt{1-x^2}.$

4. $\frac{dy}{dx}+y\cos x=\frac{1}{2}\sin 2x.$

5. $(1+y^2)dx=(\tan^{-1}y-x)\,dy.$

6. $\sqrt{a^2+x^2}\left(1-\frac{dy}{dx}\right)=x+y.$

7. $(1+x^2)\,dy+\left(xy-\frac{1}{x}\right)dx=0.$

8. $\frac{dy}{dx}+y\frac{d\phi}{dx}=\phi\frac{d\phi}{dx},$

where ϕ is a function of x alone.

16. Form, $$\frac{dy}{dx}+Py=Qy^n,$$

where P, Q, are independent of y.

Divide by y^n, and substitute $z=y^{-n+1}$. The new equation between z and x will be linear. (See Art. 14.)

17. Solve the following equations:

1. $(1-x^2)\frac{dy}{dx}-xy=axy^2.$

2. $3y^2\frac{dy}{dx}-ay^3=x+1.$

3. $\frac{dy}{dx}=2xy\,(ax^2y^2-1).$

4. $\frac{dy}{dx}(x^2y^3+xy)=1.$

5. $\frac{dy}{dx}+y\cos x=y^n\sin 2x.$

6. $(y\log x-1)\,y\,dx=x\,dy.$

7. $ax^2y^n\,dy+y\,dx=2x\,dy.$

8. $y-\cos x\frac{dy}{dx}=y^2\cos x\,(1-\sin x).$

9. $y\frac{dy}{dx}+by^2=a\cos x.$

CHAPTER III.

EXACT DIFFERENTIAL EQUATIONS AND INTEGRATING FACTORS.

18. $Mdx + Ndy$ is an exact differential when

$$\frac{dM}{dy} = \frac{dN}{dx}. \tag{1}$$

The solution of

$$Mdx + Ndy = 0, \quad \text{in this case is}$$

$$\int Mdx + \int \left(N - \frac{d}{dy}\int Mdx\right)dy = c,$$

or

$$\int Ndy + \int \left(M - \frac{d}{dx}\int Ndy\right)dx = c.$$

In integrating with respect to x, y is regarded as constant, and in integrating with respect to y, x is regarded as constant.

19. Solve the following equations after applying the condition (1) for an exact differential:

1. $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0.$

2. $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0.$

3. $\left(1 + \frac{y^2}{x^2}\right)dx - \frac{2y}{x}dy = 0.$

4. $\frac{2x\,dx}{y^3} + \left(\frac{1}{y^2} - \frac{3x^2}{y^4}\right)dy = 0.$

5. $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2+y^2} = 0.$

6. $\dfrac{dx}{\sqrt{x^2+y^2}} + \left(1 - \dfrac{x}{\sqrt{x^2+y^2}}\right)\dfrac{dy}{y} = 0.$

7. $\left(x + \dfrac{1}{\sqrt{y^2-x^2}}\right)dx + \left(y - \dfrac{x}{y\sqrt{y^2-x^2}}\right)dy = 0.$

8. $(1 + e^{\frac{x}{y}})\,dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)dy = 0.$

9. $e^x(x^2+y^2+2x)\,dx + 2ye^x\,dy = 0.$

10. $(m\,dx + n\,dy)\sin(mx+ny) = (n\,dx + m\,dy)\cos(nx+my).$

11. $\dfrac{x\,dx + y\,dy}{\sqrt{1+x^2+y^2}} + \dfrac{y\,dx - x\,dy}{x^2+y^2} = 0.$

12. $\dfrac{x^a\,dy - ayx^{a-1}\,dx}{by^2 - gx^{2a}} + x^{a-1}\,dx = 0.$

(1), $g > 0$; (2), $g < 0$ and $= -k$; (3), $g = 0$; (4), $a = 0$; (5), $b = 0$.

20. When $Mdx + Ndy$ is not an exact differential, it may sometimes be made exact by multiplying by a factor, called an *integrating factor*. The following are some of the cases where this is possible.

21. When $Mdx + Ndy$ is homogeneous, $\dfrac{1}{Mx+Ny}$ is an integrating factor. This fails when $Mx + Ny = 0$, but in that case the solution is $y = cx$.

22. Solve the following equations by means of an integrating factor:

1. $(x^2 + 2xy - y^2)\,dx = (x^2 - 2xy - y^2)\,dy.$

2. $\dfrac{dx}{x} + \dfrac{dy}{y} + 2\left(\dfrac{dx}{y} - \dfrac{dy}{x}\right) = 0.$

3. $(x^2y^2 + xy^3)\,dx - (x^3y + x^2y^2)\,dy = 0.$

4. $x^3\,dx + (3x^2y + 2y^3)\,dy = 0.$

5. $(x\sqrt{x^2+y^2} - x^2)\,dy + (xy - y\sqrt{x^2+y^2})\,dx = 0$

(See Art. 11 for other examples.)

23. Form, $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$

$\dfrac{1}{Mx - Ny}$ is an integrating factor. This fails when

$$Mx - Ny = 0,$$

but in that case the solution is $xy = c$.

Another method of solving is to put $xy = v$, and obtain an equation between x and v or between y and v. The variables can then be separated.

24. Solve the following equations by means of an integrating factor:

1. $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0.$

2. $(x^2y^2 + xy)\,y\,dx + (x^2y^2 - 1)\,x\,dy = 0.$

3. $(x^3y^3 + 1)(x\,dy + y\,dx) + (x^2y^2 + xy)(y\,dx - x\,dy) = 0.$

4. $(\sqrt{xy} - 1)\,x\,dy - (\sqrt{xy} + 1)\,y\,dx = 0.$

5. $(y + y\sqrt{xy})\,dx + (x + x\sqrt{xy})\,dy = 0.$

6. $e^{xy}(x^2y^2 + xy)(x\,dy + y\,dx) + y\,dx - x\,dy = 0.$

7. $xy\,[1 + \cot(xy)](x\,dy + y\,dx) + x\,dy - y\,dx = 0.$

25. When
$$\frac{\dfrac{dM}{dy} - \dfrac{dN}{dx}}{N} = \phi(x),$$

then $e^{\int\phi(x)dx}$ is an integrating factor.

Or, when $$\frac{\dfrac{dN}{dx}-\dfrac{dM}{dy}}{M}=\psi(y),$$

then $e^{\int\psi(y)dy}$ is an integrating factor.

26. Solve the following equations by means of an integrating factor:

1. $(x^2+y^2+2x)dx+2y\,dy=0.$

2. $(3x^2-y^2)\dfrac{dy}{dx}=2xy.$

3. $\dfrac{dy}{dx}=\dfrac{x^2+y^2}{2xy}.$

4. $[(1-y)\sqrt{1-x^2}-xy]\,dx+[1-x^2-x\sqrt{1-x^2}]\,dy=0.$

5. $(\cos x+2y\sec y\sec^2 2x)dx+(\tan 2x\sec y-\sin x\tan y)dy=0.$

6. $\sin(3x-2y)(2\,dx-dy)+\sin(x-2y)dy=0.$

7. The Linear Equation
$$\frac{dy}{dx}+Py=Q,$$
where P and Q are independent of y.

CHAPTER IV.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND DEGREE CONTAINING THREE VARIABLES.

General form, $\quad Pdx + Qdy + Rdz = 0,$

where P, Q, R, are each functions of x, y, z.

27. If the variables can be separated, solve by integrating the parts separately.

The equation is derivable from a single primitive only when the following condition is satisfied:

$$P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) = 0. \qquad (1)$$

The solution may then be effected by first solving the equation with one of the parts Pdx, Qdy, Rdz, omitted, regarding x, y, z, respectively, constant.

Omitting Rdz, for example, we solve $Pdx + Qdy = 0$, regarding z constant, and introducing instead of an arbitrary constant of integration, Z, an undetermined function of z, which must be subsequently determined so that this primitive may satisfy the given differential equation. The equation of condition for determining Z will ultimately involve only Z and z.

28. Solve the following equations after applying the condition (1) for a single primitive:

1. $\quad \dfrac{dx}{x-a} + \dfrac{dy}{y-b} + \dfrac{dz}{z-c} = 0.$

2. $\quad (x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0.$

3. $(y+z)\,dx + (z+x)\,dy + (x+y)\,dz = 0.$

4. $yz\,dx + zx\,dy + xy\,dz = 0.$

5. $(y+z)\,dx + dy + dz = 0.$

6. $ay^2z^2\,dx + bz^2x^2\,dy + cx^2y^2\,dz = 0.$

7. $zy\,dx = zx\,dy + y^2\,dz.$

8. $(y\,dx + x\,dy)(a+z) = xy\,dz.$

9. $(y+a)^2\,dx + z\,dy = (y+a)\,dz.$

10. $(y^2+yz)\,dx + (xz+z^2)\,dy + (y^2-xy)\,dz = 0.$

11. $(2x^2 + 2xy + 2xz^2 + 1)\,dx + dy + 2\,dz = 0.$

12. $(x^2y - y^3 - y^2z)\,dx + (xy^2 - x^2z - x^3)\,dy + (xy^2 + x^2y)\,dz = 0.$

CHAPTER V.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER, OF A DEGREE ABOVE THE FIRST.

In what follows, p denotes $\frac{dy}{dx}$.

29. *When the equation can be solved with respect to* p.

The different values of p constitute so many differential equations of the first degree, which must be solved separately, using the same character for the arbitrary constant in all.

If the terms of each of these separate primitives be transposed to the first member, the product of these first members placed equal to zero will be the complete primitive.

30. Solve the following equations:

1. $p^2 - 5p + 6 = 0.$
2. $x^2p^2 - a^2 = 0.$
3. $xp^2 - a = 0.$
4. $xp^2 = 1 - x.$
5. $x^2p^2 + 3xyp + 2y^2 = 0.$
6. $p(p+y) = x(x+y).$
7. $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0.$
8. $p^3 - (x^2 + xy + y^2)p^2 + (x^3y + x^2y^2 + xy^3)p - x^3y^3 = 0.$
9. $p^2 + 2py\cot x = y^2.$

31. *When the equation can be solved with respect to* y.

Differentiate, regarding p variable as well as x and y, and

substitute for dy, pdx. There will result a differential equation of the first degree between x and p. Solve this equation, and eliminate p between its primitive and the given equation.

32. Solve the following equations:

1. $x - yp = ap^2$.
2. $y = xp^2 + 2p$.
3. $(x + yp)^2 = a^2(1 + p^2)$.
4. $y = xp + p - p^2$.
5. $(y - ap)^2 = 1 + p^2$.
6. $y = ap + bp^2$.
7. $x^2 + y = p^2$.
8. $y^2 = x^2(1 + p^2)$.
9. $y = p^2 + 2p^3$.

33. *When the equation can be solved with respect to* x. Differentiate, regarding p variable as well as x and y, and substitute for dx, $\frac{dy}{p}$. There will result a differential equation of the first degree between y and p. Solve this equation, and eliminate p between its primitive and the given equation.

34. Solve the following equations:

1. $p^2y + 2px = y$.
2. $x = p + \log p$.
3. $p^2(x^2 + 2ax) = a^2$.
4. $x^2p^2 = 1 + p^2$.
5. $(x - ap)^2 = 1 + p^2$; also when $a = 1$.
6. $x = ap + bp^2$.
7. $my - nxp = yp^2$.

35. *When the equation is homogeneous with respect to* x *and* y.

Substitute $y = vx$. If the resulting equation between p and v can be solved with respect to v, the given equation comes under Art. 31 or Art. 33.

But if we can solve with respect to p, substitute for p, $v + x\dfrac{dv}{dx}$, and there will result a differential equation of the first degree between v and x.

36. Solve the following equations:

1. $xy^2(p^2+2) = 2py^3 + x^3.$
2. $(2p+1)x^{\frac{1}{2}}y = x^{\frac{3}{2}}p^2 + 2y^{\frac{3}{2}}.$
3. $4x^2 = 3(3y - px)(y + px).$
4. $ds = \left(\dfrac{y}{2x}\right)^{\frac{1}{2}} dx + \left(\dfrac{x}{2y}\right)^{\frac{1}{2}} dy$, where $ds = \sqrt{1+p^2}\cdot dx.$
5. $(nx + py)^2 = (1+p^2)(y^2 + nx^2).$

37. *Clairaut's Form,*

$$y = px + f(p).$$

The solution is immediately obtained by substituting $p = c$.

38. Solve the following equations:

1. $y = px + \dfrac{m}{p}\cdot$
2. $y = px + p - p^3.$
3. $y^2 - 2pxy - 1 = p^2(1 - x^2).$
4. $y = 2px + y^2p^3.$ Put $y^2 = y'$.
5. $ayp^2 + (2x - b)p = y.$ Put $y^2 = y'$.
6. $x^2(y - px) = yp^2.$ Put $y^2 = y'$, $x^2 = x'$.
7. $e^{3x}(p-1) + p^3e^{2y} = 0.$ Put $e^x = x'$, $e^y = y'$.
8. $(px - y)(py + x) = h^2p.$ Put $y^2 = y'$, $x^2 = x'$.

CHAPTER VI.

SINGULAR SOLUTIONS.

39. A *singular solution* of a differential equation is a solution which is not included in the complete primitive. Differential equations of the first degree have no singular solution. Those of higher degrees may have singular solutions, which may be derived either from the complete primitive, or directly from the differential equation.

40. Let $f(x, y, c) = 0$ be the complete primitive.

By differentiating, regarding c as the only variable, obtain $\dfrac{df}{dc} = 0$. If we eliminate c between this equation and the primitive, the result will be a singular solution, provided it satisfies the given differential equation.

41. Let $f(x, y, p) = 0$ be the given differential equation.

By differentiating, regarding p as the only variable, obtain $\dfrac{df}{dp} = 0$. If we eliminate p between this equation and the given differential equation, the result will be a singular solution, provided it satisfies the differential equation.

42. Derive the singular solution of the following equations, directly from the given equation, and also from the complete primitive:

1. $y = px + \dfrac{m}{p}.$

2. $y^2 - 2xyp + (1 + x^2)p^2 = 1.$

3. $p^3 - 4xyp + 8y^2 = 0.$ Put $y = z^2$.

4. $y = (x-1)p - p^2.$

5. $y(1+p^2) = 2xp.$

6. $x^2p^2 - 2(xy-2)p + y^2 = 0.$

7. $(y - xp)(mp - n) = mnp.$

DIFFERENTIAL EQUATIONS OF AN ORDER HIGHER THAN THE FIRST.

CHAPTER VII.

LINEAR DIFFERENTIAL EQUATIONS.

General Form,

$$\frac{d^n y}{dx^n} + X_1 \frac{d^{n-1} y}{dx^{n-1}} + X_2 \frac{d^{n-2} y}{dx^{n-2}} \cdots + X_{n-1} \frac{dy}{dx} + X_n y = X,$$

the coefficients $X_1, X_2, \ldots X_n$ and X being functions of x alone or constants.

43. Linear equations with *constant* coefficients and second member *zero* may be solved as follows:

Substitute in the given equation,

$$\frac{d^n y}{dx^n} = m^n, \quad \frac{d^{n-1} y}{dx^{n-1}} = m^{n-1}, \ldots\ldots \frac{dy}{dx} = m, \quad y = m^0 = 1.$$

There will result an equation of the nth degree in m, called the *auxiliary equation*. Find the n roots of this equation; these roots will determine a series of terms expressing the complete value of y as follows, viz. For each real root m_1, there will be a term $Ce^{m_1 x}$; for each pair of imaginary roots $a \pm b\sqrt{-1}$, a term $e^{ax}(A \sin bx + B \cos bx)$; each of the coefficients A, B, C, being an arbitrary constant if the corresponding root occur only once, but a polynomial $c_1 + c_2 x + c_3 x^2 \cdots + c_r x^{r-1}$, if the root occur r times.

44. *Roots of auxiliary equation, real and unequal.*
Solve the following equations:

1. $$\frac{d^2y}{dx^2} = a^2y.$$

2. $$\frac{d^2y}{dx^2} + 12y = 7\frac{dy}{dx}.$$

3. $$a\frac{d^2y}{dx^2} = \frac{dy}{dx}.$$

4. $$3\left(\frac{d^2y}{dx^2} + y\right) = 10\frac{dy}{dx}.$$

5. $$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = y.$$

6. $$ab\left(y + \frac{d^2y}{dx^2}\right) = (a^2 + b^2)\frac{dy}{dx}.$$

7. $$\frac{d^3y}{dx^3} = 4\frac{dy}{dx}.$$

8. $$\frac{d^3y}{dx^3} = \frac{d^2y}{dx^2} + 6\frac{dy}{dx}.$$

9. $$\frac{d^3y}{dx^3} = 7\frac{dy}{dx} - 6y.$$

10. $$\frac{d^4y}{dx^4} + 27y = 12\frac{d^2y}{dx^2}.$$

11. $$\frac{d^5y}{dx^5} - 2(a^2 + b^2)\frac{d^3y}{dx^3} + (a^2 - b^2)^2\frac{dy}{dx} = 0.$$

45. *Roots of auxiliary equation unequal, but not all real.*
Solve the following equations:

1. $$\frac{d^2y}{dx^2} + y = 0.$$

2. $$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0.$$

3. $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + b^2y = 0.$

(1), when $a > b$; (2), when $a < b$.

4. $\dfrac{d^2y}{dx^2} - 4ab\dfrac{dy}{dx} + (a^2 + b^2)^2y = 0.$

5. $\dfrac{d^2y}{dx^2} - 2\log a\dfrac{dy}{dx} + [1 + (\log a)^2]\,y = 0.$

6. $\dfrac{d^3y}{dx^3} + 2\dfrac{dy}{dx} = 0.$

7. $\dfrac{d^3y}{dx^3} = y.$

8. $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = 3y.$

9. $\dfrac{d^4y}{dx^4} = y.$

10. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} - 8y = 0.$

11. $\dfrac{d^4y}{dx^4} + 4a^4y = 0.$

12. $\dfrac{d^6y}{dx^6} = y.$

13. $\dfrac{d^6y}{dx^6} = -y.$

14. $\dfrac{d^8y}{dx^8} = y.$

46. *Auxiliary equation containing equal roots.*
Solve the following equations:

1. $\dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + a^2y = 0.$

2. $\dfrac{d^2y}{dx^2} = 0.$

3. $\dfrac{d^3y}{dx^3} = 4\dfrac{d^2y}{dx^2}.$

4. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4y = 0.$

5. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = 0.$

6. $\dfrac{d^4y}{dx^4} + 2n^2\dfrac{d^2y}{dx^2} + n^4y = 0.$

7. $\dfrac{d^4y}{dx^4} - 3\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} = 0.$

8. $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 14\dfrac{d^2y}{dx^2} - 20\dfrac{dy}{dx} + 25y = 0,$ the first member of auxiliary equation being a perfect square.

9. $\dfrac{d^ny}{dx^n} + \dfrac{d^{n-2}y}{dx^{n-2}} = 0.$

47. Linear Equations with constant coefficients and second member *not* zero.

There are two methods of solution:

First. Method of Variable Parameters. — Solve the equation by Art. 43, regarding the second member as zero.

Supposing it to be of the nth order, this value of y will contain n arbitrary constants. Derive from it the successive differential coefficients, $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \cdots \dfrac{d^{n-1}y}{dx^{n-1}}$; then differentiate the values of $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \cdots \dfrac{d^{n-1}y}{dx^{n-1}}$, regarding the arbitrary constants alone as variable, and place these n results equal to zero, except the last, which put equal to the second member of the given equation. These n conditions will determine expressions for the n arbitrary constants, which are to be substituted in the original expression for y.

SECOND METHOD. — By successively differentiating the given equation, obtain, either directly or by elimination, a new differential equation of a higher order with the second member zero. Solve this by Art. 43, and determine the values of the superfluous constants so as to satisfy the given differential equation. In this last work of determining the superfluous constants all the other constants may be regarded as zero.

48. Solve the following equations:

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = x.$

2. $\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = a.$

3. $\frac{d^2y}{dx^2} - a^2y = x + 1.$

4. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x.$

5. $\frac{d^4y}{dx^4} - a^4y = x^3.$

6. $\frac{d^2y}{dx^2} + n^2y = 1 + x + x^2.$

7. $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2y = e^x$; also when $a = 1$.

8. $\frac{d^2y}{dx^2} + n^2y = \cos ax$; also when $a = n$.

9. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{nx}$; also when $n = 2$, or $n = 3$.

10. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = xe^{nx}.$

11. $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = x^2e^{3x}.$

12. $\frac{d^2y}{dx^2} + 4y = x\sin^2x.$

13. $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2 \cos ax$; also when $a = 1$.

14. $\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} + 4y = e^x \cos x$.

49. Linear equations of the form

$$(a+bx)^n\frac{d^ny}{dx^n}+A_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}}\cdots+A_{n-1}(a+bx)\frac{dy}{dx}+A_ny=X,$$

where A_1, A_2, $\cdots A_n$ are constants, and X a function of x alone.

Put $a + bx = e^t$, and change the independent variable from x to t. The new differential equation between y and t will be linear with *constant* coefficients, and may be solved by Art. 47.

50. Solve the following equations:

1. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 3y$.

2. $(x+a)^2\dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$.

3. $(a+bx)^2\dfrac{d^2y}{dx^2} + b(a+bx)\dfrac{dy}{dx} + b^2y = 0$.

4. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$.

5. $(2x-1)^3\dfrac{d^3y}{dx^3} + (2x-1)\dfrac{dy}{dx} - 2y = 0$.

6. $$16(x+1)^4\frac{d^4y}{dx^4} + 96(x+1)^3\frac{d^3y}{dx^3} + 104(x+1)^2\frac{d^2y}{dx^2} + 8(x+1)\frac{dy}{dx} + y = x^2 + 4x + 3.$$

7. $x^4\dfrac{d^4y}{dx^4} + 6x^3\dfrac{d^3y}{dx^3} + 9x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1+\log x)^2$.

8. $x^2\dfrac{d^2y}{dx^2} - (2m-1)x\dfrac{dy}{dx} + (m^2+n^2)y = n^2x^m\log x$.

CHAPTER VIII.

SOME SPECIAL FORMS OF DIFFERENTIAL EQUATIONS OF HIGHER ORDERS.

51. Form, $\dfrac{d^n y}{dx^n} = X$, where X is a function of x alone.

The expression for y is found by integrating X successively n times with regard to x. Or solve by Art. 47.

52. Solve the following equations:

1. $x\dfrac{d^3 y}{dx^3} = 2.$

2. $\dfrac{d^4 y}{dx^4} = \dfrac{1}{(x+a)^2}.$

3. $\dfrac{d^n y}{dx^n} = x^m.$

4. $\dfrac{d^4 y}{dx^4} = x \cos x.$

5. $e^x \dfrac{d^4 y}{dx^4} + 4 \cos x = 0.$

6. $\dfrac{d^n y}{dx^n} = xe^x.$

7. $\dfrac{d^3 y}{dx^3} = \sin^3 x.$

53. Form, $\dfrac{d^2 y}{dx^2} = Y$, where Y is a function of y only.

Multiplying both members by $2\dfrac{dy}{dx}$, and integrating, we have $\left(\dfrac{dy}{dx}\right)^2 = 2\displaystyle\int Y dy + c_1$. Therefore $x = \displaystyle\int \frac{dy}{\left(2\int Y dy + c_1\right)^{\frac{1}{2}}} + c_2.$

54. Solve the following equations:

1. $$\frac{d^2y}{dx^2} = a^2 y.$$

2. $$\frac{d^2y}{dx^2} = -a^2 y.$$

3. $$y^3 \frac{d^2y}{dx^2} = a.$$

4. $$\frac{d^2y}{dx^2} = e^{ny}.$$

5. $$\frac{d^2y}{dx^2} = \frac{1}{\sqrt{ay}}.$$

55. *Equations not containing* y *directly.*

By assuming the differential coefficient of the lowest order in the given equation equal to z, and consequently the other differential coefficients equal to the successive differential coefficients of z with respect to x, we shall obtain a new differential equation between z and x of a lower order than the given equation.

56. Solve the following equations:

1. $$x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

2. $$\frac{d^2y}{dx^2} = a^2 + b^2\left(\frac{dy}{dx}\right)^2.$$

3. $$\frac{dy}{dx} - x\frac{d^2y}{dx^2} = f\left(\frac{d^2y}{dx^2}\right).$$

4. $$a^2\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

5. $$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^3.$$

6. $$(1 + x^2)\frac{d^2y}{dx^2} + 1 + \left(\frac{dy}{dx}\right)^2 = 0.$$

7. $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2.$

8. $2x\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2} = \left(\dfrac{d^2y}{dx^2}\right)^2 - a^2.$

9. $\dfrac{d^3y}{dx^3}\cdot\dfrac{d^2y}{dx^2} = \left(1 - \dfrac{d^3y}{dx^3}\right)\left[1 + \left(\dfrac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{2}}.$

10. $\dfrac{d^3y}{dx^3}\left(\dfrac{dy}{dx}\right)^3 = 1.$

57. *Equations not containing* x *directly.*

By assuming $\dfrac{dy}{dx} = z$, and consequently

$$\frac{d^2y}{dx^2} = z\frac{dz}{dy}, \quad \frac{d^3y}{dx^3} = z^2\frac{d^2z}{dy^2} + z\left(\frac{dz}{dy}\right)^2, \text{ etc.,}$$

changing the independent variable from x to y, we shall obtain a new differential equation between z and y of a lower order than the given equation.

58. Solve the following equations:

1. $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1.$

2. $y(1-\log y)\dfrac{d^2y}{dx^2} + (1+\log y)\left(\dfrac{dy}{dx}\right)^2 = 0.$

3. $y\dfrac{d^2y}{dx^2} + \left[\left(\dfrac{dy}{dx}\right)^2 + a^2\left(\dfrac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{2}} = \left(\dfrac{dy}{dx}\right)^2.$

4. $\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}\cdot f\left[\left(\dfrac{dy}{dx}\right)^{-1}\cdot\dfrac{d^2y}{dx^2}\right].$

5. $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2\log y.$

6. $y^2\left(\dfrac{d^2y}{dx^2}\right)^2 + n\left(\dfrac{dy}{dx}\right)^4 = y^2\dfrac{dy}{dx}\cdot\dfrac{d^3y}{dx^3} + ny\left(\dfrac{dy}{dx}\right)^2\dfrac{d^2y}{dx^2}.$

CHAPTER IX.

SIMULTANEOUS DIFFERENTIAL EQUATIONS.

59. *Simultaneous differential equations of the first order.*

There should be n given equations between $n+1$ variables. Selecting one of these for the independent variable, we may, by differentiating the given equations a sufficient number of times, eliminate all but one of the dependent variables and their differential coefficients. The resulting differential equation between two variables must be solved by the methods previously given, and from its primitive and the given equations may be obtained the values of the other dependent variables. The complete solution will consist of n equations containing n arbitrary constants.

In general, if we differentiate the given equations $n-1$ times successively, we shall have in all n^2 equations, which are just sufficient for the elimination of $n-1$ variables, together with their $n(n-1)$ differential coefficients. Shorter processes for the elimination will frequently suggest themselves in special cases.

60. Solve the following simultaneous equations:

1. $$\begin{cases} \dfrac{dx}{dt}+4x+\dfrac{y}{4}=0, \\[2ex] \dfrac{dy}{dt}+3y-x=0. \end{cases}$$

2. $$-dx=\frac{dy}{3y+4z}=\frac{dz}{2y+5z}.$$

3. $$\frac{dx}{y-7x}=\frac{-dy}{2x+5y}=dt.$$

4. $$\begin{cases} a\dfrac{dz}{dx} + n^2 y = e^x, \\[2ex] \dfrac{dy}{dx} + az = 0. \end{cases}$$

5. $$\begin{cases} \dfrac{dx}{dt} + 5x + y = e^t, \\[2ex] \dfrac{dy}{dt} + 3y - x = e^{2t}. \end{cases}$$

6. $$\frac{dx}{2y - 5x + e^t} = \frac{dy}{x - 6y + e^{2t}} = dt.$$

7. $$\begin{cases} 4\dfrac{dx}{dt} + 9\dfrac{dy}{dt} + 44x + 49y = t, \\[2ex] 3\dfrac{dx}{dt} + 7\dfrac{dy}{dt} + 34x + 38y = e^t. \end{cases}$$

8. $$\begin{cases} 4\dfrac{dx}{dt} + 9\dfrac{dy}{dt} + 11x + 31y = e^t, \\[2ex] 3\dfrac{dx}{dt} + 7\dfrac{dy}{dt} + 8x + 24y = e^{2t}. \end{cases}$$

9. $$\begin{cases} 4\dfrac{dx}{dt} + 9\dfrac{dy}{dt} + 2x + 31y = e^t, \\[2ex] 3\dfrac{dx}{dt} + 7\dfrac{dy}{dt} + x + 24y = 3. \end{cases}$$

10. $$\begin{cases} \dfrac{dx}{dt} + \dfrac{2}{t}(x - y) = 1, \\[2ex] \dfrac{dy}{dt} + \dfrac{1}{t}(x + 5y) = t. \end{cases}$$

11. $$\begin{cases} t\,dx = (t - 2x)\,dt, \\ t\,dy = (tx + ty + 2x - t)\,dt. \end{cases}$$

12. $$\begin{cases} \dfrac{dx}{dt} = ny - mz, \\ \dfrac{dy}{dt} = lz - nx, \\ \dfrac{dz}{dt} = mx - ly. \end{cases}$$

13. $$\begin{cases} lt\dfrac{dx}{dt} = mn\,(y - z), \\ mt\dfrac{dy}{dt} = nl\,(z - x), \\ nt\dfrac{dz}{dt} = lm\,(x - y). \end{cases}$$

61. *Simultaneous differential equations of an order higher than the first.*

By differentiating the given equations a sufficient number of times, we may eliminate all but one of the dependent variables and their differential coefficients, and thus obtain a differential equation between two variables, which must be solved by the appropriate methods. Its primitive, together with the given equation, will enable us to determine the values of the other dependent variables. The general solution will contain a number of arbitrary constants equal to the sum of the highest orders of differential coefficients in the several given equations.

62. Solve the following simultaneous equations:

1. $$\begin{cases} \dfrac{d^2x}{dt^2} + n^2x = 0, \\ \dfrac{d^2y}{dt^2} - n^2x = 0. \end{cases}$$

2. $$\begin{cases} \dfrac{d^2x}{dt^2} - 3x - 4y + 3 = 0, \\ \dfrac{d^2y}{dt^2} + x + y + 5 = 0. \end{cases}$$

3. $$\begin{cases} \dfrac{d^2 x}{dt^2} - 3x - 4y + 3 = 0, \\[2ex] \dfrac{d^2 y}{dt^2} + x - 8y + 5 = 0. \end{cases}$$

4. $$\begin{cases} \dfrac{d^2 x}{dt^2} + n^2 y = 0, \\[2ex] \dfrac{d^2 y}{dt^2} - n^2 x = 0. \end{cases}$$

5. $$\begin{cases} 2\dfrac{d^2 y}{dx^2} - \dfrac{dz}{dx} - 4y = 2x, \\[2ex] 2\dfrac{dy}{dx} + 4\dfrac{dz}{dx} - 3z = 0. \end{cases}$$

6. $$\begin{cases} \dfrac{d^3 y}{dx^3} + 4\dfrac{d^2 z}{dx^2} + (5 - n^2)\dfrac{dy}{dx} + 2(1 - n^2)z = 0, \\[2ex] \dfrac{d^3 z}{dx^3} + 4\dfrac{d^2 y}{dx^2} + (5 - n^2)\dfrac{dz}{dx} + 2(1 - n^2)y = 0. \end{cases}$$

7. $$\begin{cases} \dfrac{d^4 y}{dx^4} - 4\dfrac{d^3 z}{dx^3} + 4\dfrac{d^2 y}{dx^2} - y = 0, \\[2ex] \dfrac{d^4 z}{dx^4} - 4\dfrac{d^3 y}{dx^3} + 4\dfrac{d^2 z}{dx^2} - z = 0. \end{cases}$$

CHAPTER X.

GEOMETRICAL EXAMPLES.

63. Expressions involved in the examples, p representing $\frac{dy}{dx}$, and q representing $\frac{d^2y}{dx^2}$.

$$\text{Subtangent} = \frac{y}{p}. \qquad \text{Subnormal} = py.$$

$$\text{Normal} = y\sqrt{1+p^2}. \qquad \frac{ds}{dx} = \sqrt{1+p^2}.$$

$$\text{Intercept of tangent on axis of } X = x - \frac{y}{p}.$$

$$\text{Intercept of tangent on axis of } Y = y - px.$$

$$\text{Radius of curvature} = \mp \frac{(1+p^2)^{\frac{3}{2}}}{q}.$$

1. Find the curve whose subtangent varies as (is n times) the abscissa.

2. Find the curve whose subnormal is constant and equal to $2a$.

3. Find the curve whose normal is equal to the square of the ordinate.

4. Find the curve for which $s = mx^2$.

5. Find the curve for which $s^2 = y^2 - a^2$.

The *orthogonal trajectory* of a series of curves is a curve that intersects them all at right angles.

Describe the curves represented by the following equations, and find their orthogonal trajectories:

6. $y = mx$, m being the variable parameter.

7. $y^2 = 2ax - x^2$, a being the variable parameter.

8. $y^2 = 4ax$, a being the variable parameter.

9. $xy = k^2$, k being the variable parameter.

10. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, b being the variable parameter.

11. $x^2 + m^2y^2 = m^2a^2$, a being the variable parameter.

12. $\dfrac{x^2}{b^2 + h^2} + \dfrac{y^2}{b^2} = 1$, b being the variable parameter.

The three following examples require the singular solution:

13. Find the curve such that the sum of the intercepts of the tangent on the axes of X and Y is constant and equal to a.

14. Find the curve such that the part of the tangent between the axes of X and Y is constant and equal to a.

15. Find the curve such that the area of the right triangle formed by the tangent with the axes of X and Y is constant and equal to a^2.

The following examples require the solution of differential equations of the second order:

16. Find the curve such that the length of the arc measured from some fixed point of it is equal to the intercept of the tangent on the axis of X.

17. Find the curve whose radius of curvature varies as (is n times) the cube of the normal.

18. Find the curve whose radius of curvature is equal to the normal; first, when the two have the same direction; second, when they have opposite directions.

19. Find the curve whose radius of curvature is equal to twice the normal; first, when the two have the same direction; second, when they have opposite directions.

ANSWERS.

Art. 3. $\left(p = \frac{dy}{dx}.\right)$

1. $y(1+x) + px(1-y) = 0.$
2. $(x^2+1)\,pxy = y^2 + 1.$
3. $\tan x = p \tan y.$
4. $(1+x^2)\,p + y = \tan^{-1}x.$
5. $(y \log x - 1)\,y = px.$
6. $y = px + p - p^3.$
7. $xp^2 = a.$
8. $p^2 + 2py \cot x = y^2.$
9. $x^2p^2 = 1 + p^2.$

Art. 5.

1. $\frac{d^2y}{dx^2} + a^2 y = 0.$
2. $\frac{d^2y}{dx^2} - a^2 y = 0.$
3. $\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + a^2 y = 0.$
4. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 3y.$
5. $\frac{d^2y}{dx^2} + n^2 y = \cos ax.$

Art. 7.

1. $\frac{d^3y}{dx^3} = 7\frac{dy}{dx} - 6y.$
2. $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} = 3y.$
3. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x.$
4. $\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0.$
5. $\frac{d^4y}{dx^4} - a^4 y = x^3.$

Art. 9.

1. $\log(xy) + x - y = c.$
2. $\frac{x+y}{xy} + \log\frac{y}{x} = c.$
3. $(1+x^2)(1+y^2) = cx^2.$
4. $x^2 y = ce^{\frac{y}{a}}.$

5. $\log\left[(y+\sqrt{1+y^2})\sqrt{1+y^2}\right]=\dfrac{x}{\sqrt{1+x^2}}+c.$

6. $\cos y = c\cos x.$

7. $\tan x \tan y = c.$

8. $\sin^2 x + \sin^2 y = c.$

9. $xy - 1 = c(x+y+1).$

Art. 11.

1. $y = ce^{-\frac{x}{y}}.$

2. $y = ce^{-\sqrt{\frac{x}{y}}}.$

3. $y = ce^{\frac{y}{x}}.$

4. $x^2 = c^2 + 2cy.$

5. $x = ce^{-\sin\frac{y}{x}}.$

6. $(y+x)^2(y+2x)^3 = c.$

7. $\log(x^2+y^2) = 2\tan^{-1}\dfrac{x}{y} + c.$

8. $xy\cos\dfrac{y}{x} = c.$

9. (1), $\log(x^2 - mxy + y^2) + \dfrac{2m}{\sqrt{4-m^2}}\tan^{-1}\dfrac{2y-mx}{x\sqrt{4-m^2}} = c.$

(2), $x - y = ce^{\frac{x}{y-x}}.$

(3), $\dfrac{(2y - mx + x\sqrt{m^2-4})^{m-\sqrt{m^2-4}}}{(2y - mx - x\sqrt{m^2-4})^{m+\sqrt{m^2-4}}} = c.$

10. $y\sin\alpha - x\cos\alpha + \sqrt{x^2+y^2} = c(x^2+y^2).$

Art. 13.

1. $(y-x+1)^2(y+x-1)^5 = c.$

2. $x + 2y + \log(2x+y-1) = c.$

3. $x + 5y + 2 = c(x-y+2)^4.$

4. $4x - 8y = \log(4x+8y+5) + c.$

5. $\log\left[2(3x-1)^2 + (3y-5)^2\right] - \sqrt{2}\tan^{-1}\dfrac{\sqrt{2}(3x-1)}{3y-5} = c.$

Art. 15.

1. $y = cx^a + \dfrac{x}{1-a} - \dfrac{1}{a}$.

2. $y = ax + cx\sqrt{1-x^2}$.

3. $y = \dfrac{x}{\sqrt{1-x^2}} + ce^{-\frac{x}{\sqrt{1-x^2}}}$.

4. $y = \sin x - 1 + ce^{-\sin x}$.

5. $x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}$.

6. $(x + \sqrt{a^2+x^2})y = a^2 \log(x + \sqrt{a^2+x^2}) + c$.

7. $y\sqrt{1+x^2} = \log \dfrac{\sqrt{1+x^2}-1}{x} + c$.

8. $y = ce^{-\phi} + \phi - 1$.

Art. 17.

1. $y = (c\sqrt{1-x^2} - a)^{-1}$.

2. $y^3 = ce^{ax} - \dfrac{x+1}{a} - \dfrac{1}{a^2}$.

3. $y = \left[ce^{2x^2} + \dfrac{a}{2}(2x^2+1)\right]^{-\frac{1}{2}}$.

4. $x = \dfrac{e^{\frac{y^2}{2}}}{(2-y^2)e^{\frac{y^2}{2}} + c}$.

5. $y^{-n+1} = ce^{(n-1)\sin x} + 2\sin x + \dfrac{2}{n-1}$.

6. $y = (cx + \log x + 1)^{-1}$.

7. $x = \dfrac{(n+2)y^2}{ay^{n+2}+c}$.

8. $y = \dfrac{\tan x + \sec x}{\sin x + c}$.

9. $(4b^2+1)y^2 = 2a(\sin x + 2b\cos x) + ce^{-2bx}$.

Art. 19.

1. $x^4 + 6x^2y^2 + y^4 = c$.

2. $x^3 - 6x^2y - 6xy^2 + y^3 = c$.

3. $x^2 - y^2 = cx$.

4. $x^2 - y^2 = cy^3$.

5. $x^2 + y^2 + 2\tan^{-1}\dfrac{y}{x} = c$.

6. $y^2 = c^2 - 2cx$.

7. $x^2 + y^2 + 2\sin^{-1}\dfrac{x}{y} = c$.

8. $x + ye^{\frac{x}{y}} = c$.

9. $e^x(x^2+y^2)=c.$

10. $\cos(mx+ny)+\sin(nx+my)=c.$

11. $\sqrt{1+x^2+y^2}+\tan^{-1}\dfrac{x}{y}=c.$

12. (1), $\log\dfrac{x^a\sqrt{g}+y\sqrt{b}}{x^a\sqrt{g}-y\sqrt{b}}=\dfrac{2x^a\sqrt{bg}}{a}+c.$

(2), $\tan^{-1}\dfrac{y\sqrt{b}}{x^a\sqrt{k}}+\dfrac{x^a\sqrt{bk}}{a}=c.$

(3), $x^a\left(\dfrac{1}{a}-\dfrac{1}{by}\right)=c.$

(4), $\dfrac{\sqrt{g}+y\sqrt{b}}{\sqrt{g}-y\sqrt{b}}=cx^{2\sqrt{bg}}.$

(5), $\dfrac{x^a}{a}-\dfrac{y}{gx^a}=c.$

Art. 22.

1. $x^2+y^2=c(x+y).$
2. $x^2-y^2+xy=c.$
3. $y=cx.$
4. $x^2+2y^2=c\sqrt{x^2+y^2}.$
5. $y=cx.$

Art. 24.

1. $x=cye^{\frac{1}{xy}}.$
2. $y=ce^{xy}.$
3. $xy-\dfrac{1}{xy}=\log cy^2.$
4. $\dfrac{2}{\sqrt{xy}}=\log\dfrac{cx}{y}.$
5. $xy=c.$
6. $xye^{xy}=\log\dfrac{cy}{x}.$
7. $xy+\log\sin(xy)=\log\dfrac{cx}{y}.$

Art. 26.

1. $e^x(x^2+y^2)=c.$
2. $x^2-y^2=cy^3.$
3. $x^2-y^2=cx.$
4. $y\sqrt{1-x^2}+x(1-y)=c.$

5. $\sin x \cos y + y \tan 2x = c.$ 6. $\sin^2 x \sin 2(x - y) = c.$

7. $y = e^{-\int P dx}\left(\int Q e^{\int P dx}\, dx + c\right).$

Art. 28.

1. $(x - a)(y - b)(z - c) = c'.$ 7. $z = ce^{\frac{x}{y}}.$

2. $x^2 + 2y^2 - 6xy - 2xz + z^2 = c.$ 8. $xy = c(a + z).$

3. $yz + zx + xy = c.$ 9. $x = \dfrac{z}{y + a} + c.$

4. $xyz = c.$ 10. $y(x + z) = c(y + z).$

5. $e^x(y + z) = c.$ 11. $e^{x^2}(x + y + z^2) = c.$

6. $\dfrac{a}{x} + \dfrac{b}{y} + \dfrac{c}{z} = c'.$ 12. $\dfrac{y + z}{x} + \dfrac{z + x}{y} = c.$

Art. 30.

1. $(y - 2x + c)(y - 3x + c) = 0,$ or $(5x - 2y + c)^2 = x^2.$

2. $(y + c)^2 = a^2(\log x)^2.$ 3. $(y + c)^2 = 4ax.$

4. $(y + c)^2 = (\sqrt{x - x^2} + \sin^{-1}\sqrt{x})^2.$

5. $(xy + c)(x^2 y + c) = 0.$

6. $(x^2 - 2y + c)[e^x(x + y - 1) + c] = 0.$

7. $(y + c)(y + x^2 + c)(xy + cy + 1) = 0.$

8. $(x^3 - 3y + c)(e^{\frac{x^2}{2}} + cy)(xy + cy + 1) = 0.$

9. $y^2 \sin^2 x + 2cy + c^2 = 0.$

Art. 32.

1. Eliminate p by means of $x = \dfrac{p}{\sqrt{1 - p^2}}(c + a \sin^{-1} p).$

2. Eliminate p by means of $x(p - 1)^2 = \log p^2 - 2p + c.$

3. Eliminate p by means of $x = \dfrac{p}{\sqrt{1 + p^2}}\left(c + \dfrac{a}{p} + a \tan^{-1} p\right).$

4. $y = cx + c - c^2.$

5. $x = a \log (ay \pm \sqrt{a^2 + y^2 - 1}) + \log (y \mp \sqrt{a^2 + y^2 - 1}) + c.$

6. $x \pm \sqrt{a^2 + 4by} = a \log (a \pm \sqrt{a^2 + 4by}) + c.$

7. $c(2y \pm x\sqrt{x^2 + y})^{\sqrt{17}} = \dfrac{\pm 4\sqrt{x^2+y} - x(\sqrt{17} - 1)}{\pm 4\sqrt{x^2+y} + x(\sqrt{17}+1)}.$

8. $\left[y\sqrt{y^2 - x^2} - x^2 \log \dfrac{y + \sqrt{y^2 - x^2}}{x}\right]^2 = (y^2 + x^2 \log cx^2)^2.$

9. $4(x+c)^3 + (x+c)^2 - 18y(x+c) - 27y^2 - 4y = 0.$

Art. 34.

1. $y^2 = 2cx + c^2.$

2. $x + 1 = \pm\sqrt{2y+c} + \log(\pm\sqrt{2y+c} - 1).$

3. $(e^{\frac{y}{a}} - ac)^2 = 2cxe^{\frac{y}{a}}.$

4. $e^{2y} + 2cxe^y + c^2 = 0.$

5. $2y + c = \dfrac{ax^2 \pm x\sqrt{a^2 + x^2 - 1}}{a^2 - 1} + \log(x \pm \sqrt{a^2 + x^2 - 1});$

when $a = 1$, $4y = x^2 - \log(cx^2).$

6. $b^2(6y + c)^2 + (6abx + a^3)(6y + c) - 3a^2x^2 - 16bx^3 = 0.$

7. $c(nx^2 + 2y^2 \pm x\sqrt{n^2x^2 + 4my^2})^n = [(2m - n)x \pm \sqrt{n^2x^2 + 4my^2}]^{2m}.$

Art. 36.

1. $(x^2 - y^2 + c)(x^2 - y^2 + cx^4) = 0.$

2. $(\sqrt{x} - \sqrt{y} + c)(\sqrt{x} - \sqrt{y} + cx) = 0.$

3. $3x^4 + 6cxy + c^2 = 0.$

4. $(y - x)^{1 \pm \sqrt{2}} = c(\sqrt{y} + \sqrt{x})^2.$

5. $x^{2k} + 2cx^{k-1}y - c^2n = 0,$ where $k = \sqrt{\dfrac{n-1}{n}}.$

Art. 38.

1. $y = cx + \frac{m}{c}$.
2. $y = cx + c - c^3$.
3. $(y - cx)^2 = 1 + c^2$.
4. $y^2 = cx + \frac{c^3}{8}$.
5. $4y^2 = 2c(2x - b) + ac^2$.
6. $y^2 = cx^2 + c^2$.
7. $e^y = ce^x + c^3$.
8. $y^2 - cx^2 = -\frac{ch^2}{c+1}$.

Art. 42.

Complete Primitives and Singular Solutions:

1. $y = cx + \frac{m}{c}$, $\quad y^2 = 4mx$.
2. $(y - cx)^2 = 1 - c^2$, $\quad y^2 - x^2 = 1$.
3. $y = c(x - c)^2$, $\quad y = \frac{4x^3}{27}$.
4. $y = c(x - 1) - c^2$, $\quad 4y = (x - 1)^2$.
5. $y^2 - 2cx + c^2 = 0$, $\quad y^2 = x^2$.
6. $(y - cx)^2 + 4c = 0$, $\quad xy = 1$.
7. $(y - cx)(mc - n) = mnc$, $\quad \left(\frac{x}{m}\right)^{\frac{1}{2}} \pm \left(\frac{y}{n}\right)^{\frac{1}{2}} = 1$.

Art. 44.

1. $y = c_1e^{ax} + c_2e^{-ax}$.
2. $y = c_1e^{3x} + c_2e^{4x}$.
3. $y = c_1e^{\frac{x}{a}} + c_2$.
4. $y = c_1e^{3x} + c_2e^{\frac{x}{3}}$.
5. $ye^{2x} = c_1e^{x\sqrt{5}} + c_2e^{-x\sqrt{5}}$.
6. $y = c_1e^{\frac{ax}{b}} + c_2e^{\frac{bx}{a}}$.
7. $y = c_1e^{2x} + c_2e^{-2x} + c_3$.
8. $y = c_1e^{3x} + c_2e^{-2x} + c_3$.
9. $y = c_1e^{2x} + c_2e^{-3x} + c_3e^x$.
10. $y = c_1e^{3x} + c_2e^{-3x} + c_3e^{x\sqrt{3}} + c_4e^{-x\sqrt{3}}$.
11. $y = c_1e^{(a-b)x} + c_2e^{(b-a)x} + c_3e^{(a+b)x} + c_4e^{-(a+b)x} + c_5$.

Art. 45.

1. $y = c_1 \sin x + c_2 \cos x.$

2. $y = e^{3x}(c_1 \sin 2x + c_2 \cos 2x).$

3. When $a > b$, $\quad y = e^{ax}(c_1 e^{x\sqrt{a^2-b^2}} + c_2 e^{-x\sqrt{a^2-b^2}})$;

 when $a < b$, $\quad y = e^{ax}(c_1 \sin x\sqrt{b^2 - a^2} + c_2 \cos x \sqrt{b^2 - a^2}).$

4. $y = e^{2abx}[c_1 \sin (a^2 - b^2)x + c_2 \cos (a^2 - b^2)x].$

5. $y = a^x(c_1 \sin x + c_2 \cos x).$

6. $y = c_1 \sin x\sqrt{2} + c_2 \cos x\sqrt{2} + c_3.$

7. $y = c_1 e^x + e^{-\frac{x}{2}}\left(c_2 \sin \dfrac{x\sqrt{3}}{2} + c_3 \cos \dfrac{x\sqrt{3}}{2}\right).$

8. $ye^x = c_1 e^{2x} + c_2 \sin x\sqrt{2} + c_3 \cos x\sqrt{2}.$

9. $y = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x.$

10. $y = c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}} + c_3 \sin 2x + c_4 \cos 2x.$

11. $y = e^{ax}(c_1 \sin ax + c_2 \cos ax) + e^{-ax}(c_3 \sin ax + c_4 \cos ax).$

12. $$y = c_1 e^x + c_2 e^{-x} + (c_3 e^{\frac{x}{2}} + c_4 e^{-\frac{x}{2}}) \sin \frac{x\sqrt{3}}{2} + (c_5 e^{\frac{x}{2}} + c_6 e^{-\frac{x}{2}}) \cos \frac{x\sqrt{3}}{2}.$$

13. $$y = c_1 \sin x + c_2 \cos x + e^{\frac{x\sqrt{3}}{2}}\left(c_3 \sin \frac{x}{2} + c_4 \cos \frac{x}{2}\right) + e^{-\frac{x\sqrt{3}}{2}}\left(c_5 \sin \frac{x}{2} + c_6 \cos \frac{x}{2}\right).$$

14. $$y = c_1 e^x + c_2 e^{-x} + c_3 \sin x + c_4 \cos x + e^{\frac{x}{\sqrt{2}}}\left(c_5 \sin \frac{x}{\sqrt{2}} + c_6 \cos \frac{x}{\sqrt{2}}\right) + e^{-\frac{x}{\sqrt{2}}}\left(c_7 \sin \frac{x}{\sqrt{2}} + c_8 \cos \frac{x}{\sqrt{2}}\right).$$

Art. 46.

1. $y = (c_1 + c_2 x) e^{ax}.$

2. $y = c_1 + c_2 x.$

3. $y = c_1 e^{4x} + c_2 + c_3 x.$

4. $y = c_1 e^{-x} + (c_2 + c_3 x) e^{2x}.$

5. $y = c_1 e^{-x} + (c_2 + c_3 x) e^x.$

6. $y = (c_1 + c_2 x) \cos nx + (c_3 + c_4 x) \sin nx.$

7. $y = (c_1 + c_2 x + c_3 x^2) e^x + c_4.$

8. $y = e^x [(c_1 + c_2 x) \sin 2x + (c_3 + c_4 x) \cos 2x].$

9. $y = c_1 + c_2 x + c_3 x^2 \cdots + c_{n-2} x^{n-3} + c_{n-1} \sin x + c_n \cos x.$

Art. 48.

1. $y = c_1 e^{3x} + c_2 e^{4x} + \dfrac{12x + 7}{144}.$

2. $y = c_1 \sin x + c_2 \cos x + (c_3 + c_4 x) e^x + a.$

3. $y = c_1 e^{ax} + c_2 e^{-ax} - \dfrac{x + 1}{a^2}.$

4. $y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right) e^x + c_3.$

5. $y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \sin ax + c_4 \cos ax - \dfrac{x^3}{a^4}.$

6. $y = c_1 \sin nx + c_2 \cos nx + \dfrac{1 + x + x^2}{n^2} - \dfrac{2}{n^4}.$

7. $y = (c_1 + c_2 x) e^{ax} + \dfrac{e^x}{(a-1)^2};$

when $a = 1, \quad y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right) e^x.$

8. $y = c_1 \sin nx + c_2 \cos nx + \dfrac{\cos ax}{n^2 - a^2};$

when $a = n, \quad y = c_1 \sin nx + c_2 \cos nx + \dfrac{x \sin nx}{2n}.$

9. $y = c_1 e^{2x} + c_2 e^{3x} + \dfrac{e^{nx}}{n^2 - 5n + 6};$

when $n = 2, \quad y = (c_1 - x) e^{2x} + c_2 e^{3x};$

when $n = 3, \quad y = c_1 e^{2x} + (c_2 + x) e^{3x}.$

10. $y = c_1 e^x + c_2 e^{2x} + \dfrac{x e^{nx}}{n^2 - 3n + 2} - \dfrac{(2n - 3) e^{nx}}{(n^2 - 3n + 2)^2}.$

11. $y = c_1 e^{4x} + c_2 e^{5x} + \dfrac{2x^2 + 6x + 7}{4} e^{3x}.$

12. $y = \left(c_1 - \dfrac{x^2}{16}\right) \sin 2x + \left(c_2 - \dfrac{x}{32}\right) \cos 2x + \dfrac{x}{8}.$

13. $y = (c_1 + c_2 x) \sin x + (c_3 + c_4 x) \cos x$

$$+ \left[\frac{x^2}{(a^2-1)^2} - \frac{4(5a^2+1)}{(a^2-1)^4}\right] \cos ax - \frac{8ax}{(a^2-1)^3} \sin ax;$$

when $a = 1$, $y = \left(c_1 + c_2 x + \dfrac{x^3}{12}\right) \sin x$

$$+ \left(c_3 + c_4 x - \frac{x^4}{48} + \frac{3x^2}{16}\right) \cos x.$$

14. $y = c_1 e^{-2x} + \left(c_2 - \dfrac{x}{20}\right) e^x \cos x + \left(c_3 + \dfrac{3x}{20}\right) e^x \sin x.$

Art. 50.

1. $y = c_1 x^3 + \dfrac{c_2}{x}.$

2. $y = c_1 (x+a)^2 + c_2 (x+a)^3 + \dfrac{3x + 2a}{6}.$

3. $y = c_1 \sin \log (a + bx) + c_2 \cos \log (a + bx).$

4. $y = x(c_1 \sin \log x + c_2 \cos \log x + \log x).$

5. $y = (2x-1)\left[c_1 + c_2 (2x-1)^{\frac{\sqrt{3}}{2}} + c_3 (2x-1)^{-\frac{\sqrt{3}}{2}}\right].$

6. $y = [c_1 + c_2 \log (x+1)] \sqrt{x+1} + \dfrac{c_3 + c_4 \log (x+1)}{\sqrt{x+1}}$

$$+ \frac{x^2 + 52x + 51}{225}.$$

7. $y = (c_1 + c_2 \log x) \sin \log x + (c_3 + c_4 \log x) \cos \log x + (\log x)^2$
$+ 2 \log x - 3.$

8. $y = x^m (c_1 \sin \log x^n + c_2 \cos \log x^n + \log x).$

Art. 52.

1. $y = c_1 + c_2 x + c_3 x^2 + x^2 \log x.$

2. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 - (x+a)^2 \log \sqrt{x+a}.$

3. $y = c_1 + c_2 x + c_3 x^2 \cdots + c_n x^{n-1} + \dfrac{|\underline{m}\, x^{m+n}}{|\underline{m+n}}$.

4. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + x \cos x - 4 \sin x$.

5. $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + e^{-x} \cos x$.

6. $y = c_1 + c_2 x + c_3 x^2 \cdots + c_n x^{n-1} + (x - n) e^x$.

7. $y = c_1 + c_2 x + c_3 x^2 + \dfrac{7 \cos x}{9} - \dfrac{\cos^3 x}{27}$.

Art. 54.

1. $ax = \log (y + \sqrt{y^2 + c_1}) + c_2$, or $y = c'_1 e^{ax} + c'_2 e^{-ax}$.

2. $ax = \sin^{-1} \dfrac{y}{c_1} + c_2$, or $y = c_1 \sin (ax + c_2)$.

3. $(c_1 x + c_2)^2 + a = c_1 y^2$.

4. $x\sqrt{2n} = c_1 \log \dfrac{\sqrt{c_1^2 e^{ny} + 1} - 1}{\sqrt{c_1^2 e^{ny} + 1} + 1} + c_2$.

5. $3x = 2a^{\frac{1}{4}} (y^{\frac{1}{2}} - 2c_1)(y^{\frac{1}{2}} + c_1)^{\frac{1}{2}} + c_2$.

Art. 56.

1. $y = c_1 \log x + c_2$.

2. $b^2 y = \log \sec [ab(x + c_1)] + c_2$.

3. $y = \dfrac{c_1 x^2}{2} + x f(c_1) + c_2$.

4. $\dfrac{2y}{a} = c_1 e^{\frac{x}{a}} + c_1^{-1} e^{-\frac{x}{a}} + c_2$.

5. $(x + c_1)^2 + (y + c_2)^2 = a^2$.

6. $y = c_1 x + (c_1^2 + 1) \log (x - c_1) + c_2$.

7. $y = c_1 \sin^{-1} x + (\sin^{-1} x)^2 + c_2$.

8. $y = \dfrac{4}{15 c_1} (x + c_1^2 a^2)^{\frac{5}{2}} + c_2 x + c_3$.

9. $12y = w^3 + c_1 w - 6w\log w + c_2$, where $w = x + c_3$.

10. $2y\sqrt{c_1} = w\sqrt{w^2 + c_1^2} + c_1^2 \log(w + \sqrt{w^2 + c_1^2}) + c_2$, where $w = x + c_3$.

Art. 58.

1. $y^2 = x^2 + c_1 x + c_2$.

2. $\log y - 1 = \dfrac{1}{c_1 x + c_2}$.

3. $c_1 y = c_2 e^{c_1 x} - \sqrt{1 + a^2 c_1^2}$.

4. $c_1 x = \log[c_1 y + f(c_1)] + c_2$.

5. $\log y = c_1 e^x + c_2 e^{-x}$.

6. $y^{n+1} = c_1 e^{c_2 x} + c_3$.

Art. 60.

1. $\begin{cases} 2x = (2c_2 - c_1 - c_2 t)e^{-\frac{7t}{2}}, \\ y = (c_1 + c_2 t)e^{-\frac{7t}{2}}. \end{cases}$

2. $\begin{cases} y = -2c_1 e^{-x} + c_2 e^{-7x}, \\ z = c_1 e^{-x} + c_2 e^{-7x}. \end{cases}$

3. $\begin{cases} 2x = e^{-6t}[(c_1 + c_2)\sin t + (c_2 - c_1)\cos t], \\ y = e^{-6t}(c_1 \sin t + c_2 \cos t). \end{cases}$

4. $\begin{cases} y = c_1 e^{nx} + c_2 e^{-nx} + \dfrac{e^x}{n^2 - 1}, \\ az = -nc_1 e^{nx} + nc_2 e^{-nx} - \dfrac{e^x}{n^2 - 1}. \end{cases}$

5. $\begin{cases} x = (c_1 + c_2 t)e^{-4t} - \dfrac{e^{2t}}{36} + \dfrac{4e^t}{25}, \\ y = -(c_1 + c_2 + c_2 t)e^{-4t} + \dfrac{7e^{2t}}{36} + \dfrac{e^t}{25}. \end{cases}$

6. $\begin{cases} x = 2c_1 e^{-4t} - c_2 e^{-7t} + \dfrac{e^{2t}}{27} + \dfrac{7e^t}{40}, \\ y = c_1 e^{-4t} + c_2 e^{-7t} + \dfrac{7e^{2t}}{54} + \dfrac{e^t}{40}. \end{cases}$

7. $$\begin{cases} x = c_1 e^{-t} + c_2 e^{-6t} - \dfrac{29\,e^t}{7} + \dfrac{19\,t}{3} - \dfrac{56}{9}, \\[2ex] y = -c_1 e^{-t} + 4\,c_2 e^{-6t} + \dfrac{24\,e^t}{7} - \dfrac{17\,t}{3} + \dfrac{55}{9}. \end{cases}$$

8. $$\begin{cases} x = (c_1 + c_2 t)\,e^{-4t} - \dfrac{49\,e^{2t}}{36} + \dfrac{31\,e^t}{25}, \\[2ex] y = -(c_1 + c_2 + c_2 t)\,e^{-4t} + \dfrac{19\,e^{2t}}{36} - \dfrac{11\,e^t}{25}. \end{cases}$$

9. $$\begin{cases} x = (c_1 \sin t + c_2 \cos t)\,e^{-4t} + \dfrac{31\,e^t}{26} - \dfrac{93}{17}, \\[2ex] y = [(c_2 - c_1)\sin t - (c_2 + c_1)\cos t]\,e^{-4t} - \dfrac{2\,e^t}{13} + \dfrac{6}{17}. \end{cases}$$

10. $$\begin{cases} x = c_1 t^{-4} + 2\,c_2 t^{-3} + \dfrac{3\,t}{10} + \dfrac{t^2}{15}, \\[2ex] y = -c_1 t^{-4} - c_2 t^{-3} - \dfrac{t}{20} + \dfrac{2\,t^2}{15}. \end{cases}$$

11. $$\begin{cases} x = c_1 t^{-2} + \dfrac{t}{3}, \\[2ex] y = c_2 e^t - c_1 t^{-2} - \dfrac{t}{3}. \end{cases}$$

12. $$\begin{cases} x = a_1 \sin kt + a_2 \cos kt + a_3, \\ y = b_1 \sin kt + b_2 \cos kt + b_3, \\ z = c_1 \sin kt + c_2 \cos kt + c_3, \end{cases}$$

where $k^2 = l^2 + m^2 + n^2$.

The arbitrary constants are connected by the following equations:

$$\frac{mc_1 - nb_1}{a_2} = \frac{na_1 - lc_1}{b_2} = \frac{lb_1 - ma_1}{c_2} = k,$$

$$la_1 + mb_1 + nc_1 = 0, \qquad \frac{a_3}{l} = \frac{b_3}{m} = \frac{c_3}{n}$$

13. $$\begin{cases} x = a_1 \sin(k\log t) + a_2 \cos(k\log t) + a_3, \\ y = b_1 \sin(k\log t) + b_2 \cos(k\log t) + b_3, \\ z = c_1 \sin(k\log t) + c_2 \cos(k\log t) + c_3, \end{cases}$$
where $k^2 = l^2 + m^2 + n^2$.

The arbitrary constants are connected by the following equations:

$$\frac{mn(c_1 - b_1)}{la_2} = \frac{nl(a_1 - c_1)}{mb_2} = \frac{lm(b_1 - a_1)}{nc_2} = k,$$

$$l^2 a_1 + m^2 b_1 + n^2 c_1 = 0, \qquad a_3 = b_3 = c_3.$$

Art. 62.

1. $$\begin{cases} x = c_1 \sin nt + c_2 \cos nt, \\ y = c_3 + c_4 t - x. \end{cases}$$

2. $$\begin{cases} x = (c_1 + c_2 t)\, e^t + (c_3 + c_4 t)\, e^{-t} - 23, \\ -2y = (c_1 - c_2 + c_2 t) e^t + (c_3 + c_4 + c_4 t) e^{-t} - 36. \end{cases}$$

3. $$\begin{cases} x = 4c_1 e^{2t} + 4c_2 e^{-2t} + c_3 e^{t\sqrt{7}} + c_4 e^{-t\sqrt{7}} + \dfrac{1}{7}, \\ y = c_1 e^{2t} + c_2 e^{-2t} + c_3 e^{t\sqrt{7}} + c_4 e^{-t\sqrt{7}} + \dfrac{9}{14}. \end{cases}$$

4. $$\begin{cases} x = e^{\frac{nt}{\sqrt{2}}}\left(c_1 \sin\dfrac{nt}{\sqrt{2}} + c_2 \cos\dfrac{nt}{\sqrt{2}}\right) + e^{-\frac{nt}{\sqrt{2}}}\left(c_3 \sin\dfrac{nt}{\sqrt{2}} + c_4 \cos\dfrac{nt}{\sqrt{2}}\right), \\ y = e^{\frac{nt}{\sqrt{2}}}\left(c_2 \sin\dfrac{nt}{\sqrt{2}} - c_1 \cos\dfrac{nt}{\sqrt{2}}\right) + e^{-\frac{nt}{\sqrt{2}}}\left(-c_4 \sin\dfrac{nt}{\sqrt{2}} + c_3 \cos\dfrac{nt}{\sqrt{2}}\right) \end{cases}$$

5. $$\begin{cases} y = (c_1 + c_2 x)\, e^x + 3c_3 e^{-\frac{3x}{2}} - \dfrac{x}{2}, \\ z = 2(3c_2 - c_1 - c_2 x) e^x - c_3 e^{-\frac{3x}{2}} - \dfrac{1}{3}. \end{cases}$$

6. $y = u + v, \quad z = -u + v,$

where $u = c_1 e^{2x} + (c_2 e^{nx} + c_3 e^{-nx}) e^x,$

$v = c_4 e^{-2x} + (c_5 e^{nx} + c_6 e^{-nx}) e^{-x}.$

7. $y = u + v, \quad z = u - v,$

where $\quad u = (c_1 + c_2 x + c_3 e^{x\sqrt{2}} + c_4 e^{-x\sqrt{2}}) e^x,$

$\quad v = (c_5 + c_6 x + c_7 e^{x\sqrt{2}} + c_8 e^{-x\sqrt{2}}) e^{-x}.$

Art. 63.

1. $x = cy^n.$

2. $y^2 = 4ax + c,$ a parabola.

3. $\pm(x + c) = \log(y + \sqrt{y^2 - 1}),$

or $\quad y = \frac{1}{2}(e^{x+c} + e^{-x-c}),$ a catenary.

4. $4my + c = 2mx\sqrt{4m^2x^2 - 1} + \log(2mx - \sqrt{4m^2x^2 - 1}).$

5. $\pm(x + c) = a\log(y + \sqrt{y^2 - a^2}),$

or $\quad \pm(x + c) = a\log\dfrac{y + \sqrt{y^2 - a^2}}{a},$

from which $\quad y = \dfrac{a}{2}\left(e^{\frac{x+c}{a}} + e^{-\frac{x+c}{a}}\right),$ a catenary.

6. $x^2 + y^2 = c^2,$ a circle.

7. $x^2 + y^2 - 2cy = 0,$ a circle.

8. $2x^2 + y^2 = 2c^2,$ an ellipse.

9. $x^2 - y^2 = c^2,$ an equilateral hyperbola.

10. $y^2 + x^2 = a^2 \log x^2 + c.$

11. $y = cx^{m^2}.$

12. $\dfrac{x^2}{h^2 - c^2} - \dfrac{y^2}{c^2} = 1,$ an ellipse or hyperbola.

13. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}},$ a parabola.

14. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$ a hypocycloid.

15. $2xy = a^2,$ an equilateral hyperbola.

16. $c^2y^2 - \log y^2 = 4c(x + c').$

17. $cy^2 - \frac{c^2}{n}(x + c')^2 = 1$, a hyperbola, when $n > 0$; an ellipse or hyperbola, when $n < 0$.

18. First, $(x + c')^2 + y^2 = c^2$, a circle.

Second, $\pm(x + c') = c \log(y + \sqrt{y^2 - c^2})$,

or $\pm(x + c') = c \log \frac{y + \sqrt{y^2 - c^2}}{c}$,

from which $y = \frac{c}{2}\left(e^{\frac{x+c'}{c}} + e^{-\frac{x+c'}{c}}\right)$, a catenary.

19. First, $x + c' = c \operatorname{vers}^{-1}\frac{y}{c} - \sqrt{2cy - y^2}$, a cycloid

Second, $(x + c')^2 = 2cy - c^2$, a parabola.

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